Model Answer / Suggestive Answer

M.A. / M.Sc (Third Semester) Examination - 2013

Mathematics (Fluid Mechanics - I)

AS - 2230

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1 (i) Stream line :- Stream is the curve drawn on the fluid surface s.t. tangent at any point is along the direction of motion. Let $\vec{q}$ is the velocity of fluid and $\frac{d\vec{r}}{dt}$ be tangent vector. Then

$$\vec{q} \times \frac{d\vec{r}}{dt} = 0 \Rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

(ii) (b)

(iii) (c)

(iv) steady

(v) (a) or (c)

(vi) (a)

(vii) (a)

(viii) Complex Potential :- Let $w = \phi + i\psi$ be taken as a function of $x+iy$ i.e. z. Suppose that $w = f(z)$ i.e. $\phi + i\psi = f(x+iy) \longrightarrow (1)$

Diff. (1) w.r.t. x and y, we get

$$\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = f'(x+iy) \longrightarrow (2)$$

and $\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} = i f'(x+iy)$

or $\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} = i\left\{\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right\}$ (by (2))

Equating real and imaginary parts, we get

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$ (which are C R - equation)

Then w is an analytic function of z and is known as the complex potential.

P.T.O.

(2)

(ix) <u>Source and Sink:-</u> If the motion of fluid consisting of symmetrical radial flow in all directions from a point that point is known as Source. If the motion of fluid consisting of symmetrical radial flow in all directions coming forward a point is known as the sink.

Source   Sink.

(x) doublet.

<u>Ques. 2:-</u> <u>Equation of continuity by the Lagrangian Method:-</u>

Total mass of the fluid with region R_0 at $t=0$

$$= \iiint_{R_0} \rho_0 \, \delta a \, \delta b \, \delta c$$

and the total mass of the fluid with region R at $t=t$

$$= \iiint_R \rho \, \delta x \, \delta y \, \delta z.$$

Then by law of conservation of mass

$$\iiint_{R_0} \rho_0 \, \delta a \, \delta b \, \delta c = \iiint_R \rho \, \delta x \, \delta y \, \delta z \longrightarrow (1)$$

Now, by change of order of integration

$$\delta x \, \delta y \, \delta z = J \, \delta a \, \delta b \, \delta c \longrightarrow (2)$$

where $J = \dfrac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix} \longrightarrow (3)$

from (1) and (2) $\iiint_{R_0} \rho_0 \, \delta a \, \delta b \, \delta c = \iiint_{R_0} J \rho \, \delta a \, \delta b \, \delta c.$

$$\Rightarrow \iiint_{R_0} (\rho_0 - \rho J) \, \delta a \, \delta b \, \delta c = 0$$

But $\delta a \, \delta b \, \delta c \neq 0$ then $\boxed{\rho_0 - \rho J = 0}$

P.T.O.

Ques. 3 The motion of a fluid is said to be irrotational when the vorticity vector Ω of every fluid particle is zero. When the vorticity vector is different from zero, the motion is said to be rotational. i.e. curl $q = 0$ for irrotational.

Given that $u = \dfrac{-2xyz}{(x^2+y^2)^2}$, $v = \dfrac{(x^2-y^2)z}{(x^2+y^2)^2}$, $w = \dfrac{y}{x^2+y^2}$

Now

$$\frac{\partial u}{\partial x} = \frac{-2yz(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{-2yz(3x^2-y^2)}{(x^2+y^2)^3}, \quad \frac{\partial w}{\partial z} = 0.$$

$$\because \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Hence it is possible liquid motion.

Again. curl $q = 0$

$\therefore$ Motion is irrotational.

Ques. 4:- Let v' be the velocity at a distance r' at any time t and ~~let~~ p be the pressure there.

Then equation of continuity is due to spherical symmetry

$$r'^2 v' = f(t) = r^2 v \longrightarrow (1)$$

so that $\dfrac{\partial v'}{\partial t} = \dfrac{f'(t)}{r'^2}$

eqn of motion

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$$\Rightarrow \frac{f'(t)}{r'^2} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

Integrating w.r.t. r'

$$-\frac{f'(t)}{r'} + \frac{1}{2}v'^2 = A - p/\rho$$

when $r' = \infty$, $v' = 0$ then $p = \Pi$ $\therefore A = \Pi/\rho$

Hence $-\dfrac{f'(t)}{r'} + \dfrac{1}{2}v'^2 = \dfrac{\Pi - p}{\rho} \longrightarrow (2)$

P.T.O.

④

Let r be the radius of cavity at any time t and v be the velocity there ($p=0$ as it is hollow)

$$\therefore -\frac{f'(t)}{r} + \tfrac{1}{2}v^2 = \frac{\Pi}{\rho}$$

$$\Rightarrow -\frac{f'(t)}{r} + \frac{1}{2}\frac{f^2(t)}{r^4} = \Pi/\rho$$

Multiply by $2f(t)\,dt$ and integrating

$$-\frac{f^2(t)}{r} = B + \frac{2\Pi}{3\rho}r^3$$

$$\Rightarrow B = -\frac{2\Pi}{3\rho}a^3 \quad (\text{initially } r=a \text{ when } v=0,\ f(t)=0)$$

$$\Rightarrow \frac{f^2(t)}{r} = \frac{2\Pi}{3\rho}(a^3-r^3)$$

$$\Rightarrow v^2 r^3 = \frac{2\Pi}{3\rho}(a^3-r^3) \Rightarrow v = \frac{dr}{dt} = -\sqrt{\frac{2\Pi}{3\rho}}\,\frac{(a^3-r^3)^{1/2}}{r^{3/2}}$$

$$dt = -\sqrt{\frac{3\rho}{2\Pi}}\cdot\frac{r^{3/2}}{(a^3-r^3)^{1/2}}\,dr$$

$$\Rightarrow t = -\int_a^0 \sqrt{\frac{3\rho}{2\Pi}}\cdot\frac{r^{3/2}\,dr}{(a^3-r^3)^{1/2}} = \frac{2a}{3}\sqrt{\frac{3\rho}{2\Pi}}\,\frac{\Gamma 5/6\ \Gamma 1/2}{2\cdot 1/3\cdot\Gamma 1/3} \longrightarrow ③$$

$$\because \Gamma n\cdot\Gamma n+\tfrac{1}{2} = \frac{\sqrt{\pi}\cdot\Gamma 2n}{2^{2n-1}} \text{ and } \Gamma n\cdot\Gamma 1-n = \frac{\pi}{\sin n\pi}$$

$$\therefore \Gamma 5/6 = \frac{2^{4/3}\,\pi\sqrt{\pi}}{\sqrt{3}\cdot(\Gamma 1/3)^2} \longrightarrow ④$$

from ③ and ④ $t = \pi^2 a\cdot 2^{5/6}\sqrt{\rho/\Pi}\,\{\Gamma 1/3\}^{-3}$.

Ques. 5:– The Energy Equation :–

Statement:– The rate of change of total energy (kinetic, potential and intrinsic) of any portion of a compressible inviscid fluid as it moves about is equal to the rate at which work is being done by the pressure on the boundary. The potential due to the external forces is supposed to be independent of time.

Proof:– Let S be the small closed surface in the region occupied by the fluid and V be the volume of the fluid. Let q be the velocity of P. Then the Euler's equation of motion is

$$\frac{dq}{dt} = f - \frac{1}{\rho}\nabla p \Rightarrow \rho.\frac{dq}{dt} = \rho f - \nabla p \longrightarrow (1)$$

A force potential Ω which is independent of time. Thus

$$f = -\nabla\Omega \text{ and } \frac{\partial\Omega}{\partial t} = 0 \longrightarrow (2)$$

from (1) and (2), we get

$$\rho\frac{dq}{dt} = -\rho\nabla\Omega - \nabla p$$

multiply by q both side

$$q.\left(\rho\frac{dq}{dt}\right) = q.(-\rho\nabla\Omega) - q.\nabla p$$

$$\Rightarrow \rho\left[\frac{d}{dt}\left(\tfrac{1}{2}q^2\right) + q.\nabla\Omega\right] = -q.\nabla p \longrightarrow (3)$$

But $\frac{d\Omega}{dt} = (q.\nabla)\Omega$ (from (2))

$\therefore$ (3) becomes $\rho\left[\frac{d}{dt}\left(\frac{1}{2}q^2\right) + \frac{d\Omega}{dt}\right] = -q.\nabla p$

$$\Rightarrow \rho\frac{d}{dt}\left(\frac{1}{2}q^2 + \Omega\right) = -q.\nabla p$$

Integrating w.r.t. V

$$\int_V \rho\frac{d}{dt}\left(\tfrac{1}{2}q^2 + \Omega\right)dv = -\int_V (q.\nabla p)\,dv$$

$$\Rightarrow \frac{d}{dt}\left[\int_V \frac{1}{2}\rho q^2 dv + \int_V \rho\Omega\, dv\right] = -\int_V (q.\nabla p)\,dv \longrightarrow (4)$$

where in we have used the equation of continuity,

$$\frac{d}{dt}(\rho\, dv) = 0 \longrightarrow (5)$$

Let T, W and I denote the kinetic, potential and energies.

Then by defn. $T = \int_V \frac{1}{2}\rho q^2 dv$, $W = \int_V \rho\Omega\, dv$, $I = \int_V \rho E\, dv \longrightarrow (6)$

$$\text{R.H.S. of (4)} = -\int_V (q.\nabla p)\,dv = -\int_V (\nabla.(pq) - p\nabla.q)\,dv$$

$$= -\int_V \nabla.(pq)\,dv + \int_V p\nabla.q\, dv$$

$$= -\int_S pq.n\, ds + \int_V p\nabla.q\, dv \longrightarrow (7)$$

we know that $\int_V p\nabla.q\, dv = -\frac{dI}{dt} \longrightarrow (8)$

Again the rate of work done by the fluid pressure on an element δs or s is $p\,\delta s\, n.q$

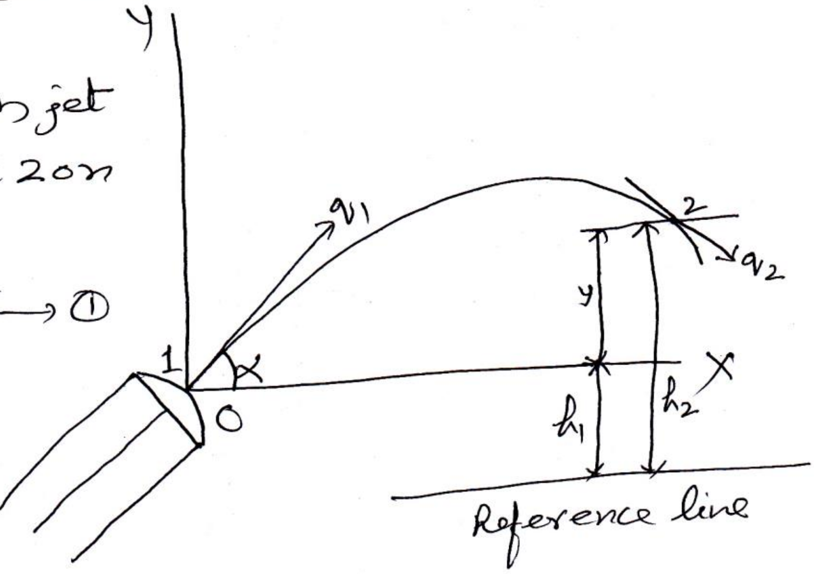
Hence the net rate at which work is being done by the fluid pressure is $\int_S pq.n\,ds = R$ (say) $\longrightarrow (9)$

using (8) and (9), eqn (7) reduces to

$$-\int_V (q\cdot\nabla p)\,dV = -R - \frac{dI}{dt} \longrightarrow (10)$$

Hence $\boxed{\frac{d}{dt}(T+\omega+I) = R}$

Ques. 6:— Trajectory of free jet:-



Bernoulli's equation between jet exist 1 and an arbitrary point 2 on the stream line

$$gh_1 + \frac{1}{2}q_1^2 = gh_2 + \frac{1}{2}q_2^2 \longrightarrow (1)$$

and $q_2^2 = q_1^2 - 2gy \longrightarrow (2)$

where $y = h_2 - h_1$.

Let $Q = q_1 S$ so that

$q_1 = \frac{Q}{S}$ Then (2) reduces to

$$q_2^2 = \left(\frac{Q}{S}\right)^2 - 2gy \longrightarrow (3)$$

consider the motion of jet along horizontal and vertical lines $\frac{dx}{dt} = q_1\cos\alpha \longrightarrow (4)$, $\frac{dy}{dt} = q_1\sin\alpha - gt \longrightarrow (5)$

Integrating w.r.t. t we get

$$x = q_1\cos\alpha\, t + c_1 \longrightarrow (6),\quad y = q_1\sin\alpha\, t - \tfrac{1}{2}gt^2 + c_2 \longrightarrow (7)$$

Initially at point 1, $t=0$, $x=0$, $y=0$ so that $c_1 = 0$ and $c_2 = 0$

Then (6) and (7) becomes

$$x = q_1\cos\alpha\, t \longrightarrow (8),\quad y = q_1\sin\alpha\, t - \tfrac{1}{2}gt^2 \longrightarrow (9)$$

Eliminating t from (8) and (9)

$$y = x\tan\alpha - \frac{1}{2}g\left(\frac{S}{Q}\right)^2\sec^2\alpha\cdot x^2 \longrightarrow (10)$$

Putting the value of y in (3)

$$q_2 = \left(\frac{Q}{S}\right)^2 - 2gx\tan\alpha + g^2x^2\sec^2\alpha\,(S/Q)^2.$$

$$(x-a)^2+y^2$$

$$\Rightarrow \phi = \frac{1}{2}\log\{(x+a)^2+y^2\} - \frac{1}{2}\log\{(x-a)^2+y^2\}$$

Diff. partially w.r.t. x and y, we get

$$\therefore u = -\frac{\partial\phi}{\partial x} = -\frac{x+a}{(x+a)^2+y^2} + \frac{x-a}{(x-a)^2+y^2} \longrightarrow (1)$$

and
$$v = -\frac{\partial\phi}{\partial y} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \longrightarrow (2)$$

$$\frac{\partial u}{\partial x} = -\frac{y^2-(x+a)^2}{\{(x+a)^2+y^2\}^2} + \frac{y^2-(x-a)^2}{[(x-a)^2+y^2]^2} \longrightarrow (3)$$

$$\frac{\partial v}{\partial y} = -\frac{(x+a)^2-y^2}{[(x+a)^2+y^2]^2} + \frac{(x-a)^2-y^2}{[(x-a)^2+y^2]^2} \longrightarrow (4)$$

Adding (3) and (4), we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Now ϕ and ψ satisfy C.R.-eqn i.e.

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \longrightarrow (5)$$

from (1) and (5),

$$\frac{\partial\psi}{\partial y} = \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}$$

integrating w.r.t. y

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} + f(x) \longrightarrow (6)$$

$$\therefore \frac{\partial\psi}{\partial x} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} + f'(x) \longrightarrow (7)$$

Again from (2) and (5)

$$\frac{\partial\psi}{\partial x} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \longrightarrow (8)$$

from (7) and (8), we get $f'(x) = 0 \Rightarrow f(x) = \text{constant}$

from (6) omitting constant

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} = \tan^{-1}\left(\frac{-2ay}{x^2+y^2-a^2}\right).$$

 Hence $\psi = \text{constant}$

i.e. $x^2+y^2-cy=a^2$

Again $w = \phi + i\psi$

$$= \tfrac{1}{2}\log[(x+a)^2+y^2] - \tfrac{1}{2}\log[(x-a)^2+y^2] + i\tan^{-1}\frac{y}{x+a} - i\tan^{-1}\frac{y}{x-a}$$

$$= \log[(x+a)+iy] - \log[(x-a)+iy]$$

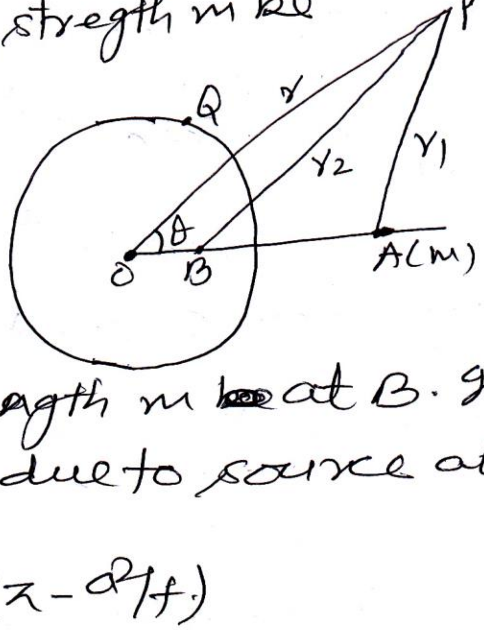
$$= \log(z+a) - \log(z-a)$$

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{1}{z+a} - \frac{1}{z-a}\right| = \frac{2a}{|z+a||z-a|} = \frac{2a}{rr'}$$

Hence $\frac{2a}{rr'} = \text{constt.} \Rightarrow rr' = \text{constt.}$

Ques 8:- Image in two dimension:- If in a liquid curve C can be drawn across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this curves is known as the image of the system with regards to the curve.

Image of source with regards to a circle:-



Let a two dimensional source of stregth m be placed at a point A. Let OA = f and B be inverse point of A with respect to the circle. Then $OA.OB = a^2$ so that $OB = \frac{a^2}{f}$.

Let there be a source of streagth m at B. If w be the complex potential due to source at A and B, then

$$w = -m\log(z-f) - m\log(z-a^2/f)$$

$$= -m\log[r\cos\theta - f + ir\sin\theta] - m\log[r\cos\theta - \tfrac{a^2}{f} + ir\sin\theta]$$

But $w = \phi + i\psi$ and equating real part, we get

$$\therefore \phi = -\frac{m}{2}\left[\log\{(r\cos\theta - f)^2 + r^2\sin^2\theta\} + \log\left\{\left(r\cos\theta - \frac{a^2}{f}\right)^2 + r^2\sin^2\theta\right\}\right]$$

$$= -\frac{m}{2}\left[\log(r^2+f^2-2rf\cos\theta) + \log\left(r^2+\frac{a^4}{f^2} - \frac{2a^2 r}{f}\cos\theta\right)\right]$$

$$\frac{\partial\phi}{\partial r} = -\frac{m}{2}\left[\frac{2(r - f\cos\theta)}{r^2+f^2-2rf\cos\theta} + \frac{2\left(r - \frac{a^2}{f}\cos\theta\right)}{r^2+\frac{a^4}{f^2} - \frac{2a^2 r\cos\theta}{f}}\right]$$

Hence the normal velocity at any point Q on the circle $= -\left(\frac{\partial\phi}{\partial r}\right)_{r=a}$

$$= \frac{m}{a}$$

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